

INTERPOLATING BETWEEN CONSTRAINED LI-YAU AND CHOW-HAMILTON HARNACK INEQUALITIES FOR A NONLINEAR PARABOLIC EQUATION

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ABSTRACT. We establish a one-parameter family of Harnack inequalities connecting the constrained trace Li-Yau differential Harnack inequality for a nonlinear parabolic equation to the constrained trace Chow-Hamilton Harnack inequality for this nonlinear equation with respect to evolving metrics related to Ricci flow on a 2-dimensional closed manifold. This result can be regarded as a nonlinear version of the previous work of Y. Zheng and the author (Arch. Math. 94 (2010), 591-600).

1. Introduction

Let $(M^2, g(t))$, $t \in [0, T)$, be a solution to the ε -Ricci flow on a 2-dimensional closed manifold M^2 . In this paper, we will establish an interpolation between the constrained trace Li-Yau differential Harnack inequality for a nonlinear parabolic equation with respect to static metrics and the constrained trace Chow-Hamilton Harnack inequality for the nonlinear parabolic equation with respect to evolving metrics related to Ricci flow. More precisely, given any nonnegative constant ε , we say that $g(t)$ is a solution to the ε -Ricci flow on a surface M^2 if

$$(1.1) \quad \frac{\partial}{\partial t} g_{ij} = -\varepsilon R \cdot g_{ij},$$

where R is the scalar curvature of $g(t)$. When $\varepsilon = 1$, the ε -Ricci flow becomes the Ricci flow. Along the ε -Ricci flow, we have

$$(1.2) \quad \frac{\partial R}{\partial t} = \varepsilon(\Delta R + R^2).$$

Using the maximum principle, one can see that $R \geq c$ for some $c \in \mathbb{R}$ is preserved along the ε -Ricci flow. Under the ε -Ricci flow, in this paper we shall study the Harnack inequalities for the following nonlinear parabolic equation

$$(1.3) \quad \frac{\partial f}{\partial t} = \Delta f - f \ln f + \varepsilon R f,$$

where Δ is the Laplacian, evolved by the ε -Ricci flow. Using the maximum principle, one can see that the solutions to the nonlinear equation (1.3) will remain positive along the ε -Ricci flow.

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The motivation to study the nonlinear parabolic equation (1.3) under the ε -Ricci flow comes from the study of expanding Ricci solitons, which has been nicely explained in [4]. The gradient expanding Ricci solitons of the Ricci flow, which arise from the singularity analysis of the Ricci flow, are defined to be complete manifolds (M, g) that the following equation

$$(1.4) \quad R_{ij} + \nabla_i \nabla_j w = c g_{ij},$$

holds for some Ricci soliton potential w and negative constant c . Taking the trace of both sides of (1.4) yields

$$(1.5) \quad R + \Delta w = cn.$$

Using the contracted Bianchi identity,

$$(1.6) \quad R - 2cw + |\nabla w|^2 = \text{constant}.$$

From (1.5) and (1.6), by choosing a proper constant, we conclude that

$$(1.7) \quad |\nabla w|^2 = \Delta w - |\nabla w|^2 - R + 4cw.$$

Recall that the Ricci flow solution for a complete gradient Ricci soliton (see Theorem 4.1 in [10]) is the pull back by $\varphi(t)$ of g up to the scale factor $c(t)$:

$$g(t) = c(t) \cdot \varphi(t)^* g,$$

where $c(t) := -2ct + 1 > 0$ and $\varphi(t)$ is the 1-parameter family of diffeomorphisms generated by $\frac{1}{c(t)} \nabla_g w$. Then the corresponding Ricci soliton potential $\varphi(t)^* w$ satisfies

$$\frac{\partial}{\partial t} \varphi(t)^* w = |\nabla \varphi(t)^* w|^2.$$

Note that along the Ricci flow, (1.7) becomes

$$|\nabla \varphi(t)^* w|^2 = \Delta \varphi(t)^* w - |\nabla \varphi(t)^* w|^2 - R + \frac{4c}{c(t)} \cdot \varphi(t)^* w.$$

Hence the Ricci soliton potential $\varphi(t)^* w$ satisfies the evolution equation

$$\frac{\partial \varphi(t)^* w}{\partial t} = \Delta \varphi(t)^* w - |\nabla \varphi(t)^* w|^2 - R + \frac{4c}{c(t)} \cdot \varphi(t)^* w.$$

If we let $\varphi(t)^* w = -\ln \tilde{f}$, this equation becomes

$$(1.8) \quad \frac{\partial \tilde{f}}{\partial t} = \Delta \tilde{f} + R \tilde{f} + \frac{4c}{c(t)} \cdot \tilde{f} \ln \tilde{f}.$$

Notice that (1.8) and (1.3) with $\varepsilon = 1$ are closely related and only differ by their last terms.

Indeed, some years ago, L. Ma [19] proved local gradient estimates for positive solutions to the elliptic equation

$$(1.9) \quad \Delta f - af \ln f - bf = 0,$$

where a and b are real constants, on a complete manifold with respect to static metrics. Again, we point out that equation (1.9) is also related to Ricci solitons. In fact, using (1.5) and (1.6), we deduce that

$$\Delta w - |\nabla w|^2 + 2cw = \text{constant}.$$

So equation (1.9) can be achieved by letting $w = -\ln f$. Later Y. Yang [29] derived local gradient estimates for positive solutions to the corresponding nonlinear parabolic equation

$$(1.10) \quad \frac{\partial f}{\partial t} = \Delta f - af \ln f - bf$$

on a static complete manifold (see also [6], [15], [25], [26]). Recently, Yang's result has been generalized by L. Ma [20, 21]. We also note that in [14], S.-Y. Hsu proved local gradient estimates for the nonlinear parabolic equation (1.10) with respect to evolving metrics related to Ricci flow. Her result is very similar to the Yang's gradient estimates [29] for the static metric case. In [4], X. Cao and Z. Zhang derived a differential Harnack inequality for equation (1.3) under the Ricci flow on any dimensional Riemannian manifold. When the dimension of the manifold is two, the author [27] improved their result.

It is well known that the study of differential Harnack inequalities originated with the work of P. Li and S.-T. Yau [17] for positive solutions of heat equations. From then on, their Harnack inequalities are often called Li-Yau differential Harnack inequalities. More importantly, Li-Yau techniques were then employed by R. Hamilton, who proved Harnack inequalities for geometric evolution equations, especially the case of the Ricci flow [13]. At present, there are a large number of Harnack inequalities for various evolution equations and their applications. The interested reader can consult the book [10] and the recent survey [23].

On the other hand, differential Harnack inequalities for (backward) heat equations coupled with the Ricci flow have become an important object, which were first studied by R. Hamilton [12]. One of the excellent important work is that G. Perelman [24] derived differential Harnack inequalities for the fundamental solution to the conjugate heat equation coupled with the Ricci flow without any curvature assumption. Later X. Cao [2], and S.-L. Kuang and Qi S. Zhang [16] both extended Perelman's result to the case of all *positive* solutions to the conjugate heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature. Besides the above work, there were also many research papers (see for example [1], [3], [5], [8], [11], [18], [28] and [30]).

In order to make a clear statement of our Harnack inequalities, we need to recall some known results, which are more or less related to our results. In [9], B. Chow and R. Hamilton extended Li-Yau differential Harnack inequality [17] for the heat equation on a closed manifold, which they called a constrained trace Harnack inequality.

Theorem A (Chow-Hamilton [9]). *Let M^n be a closed manifold with nonnegative Ricci curvature. If S and T are two solutions to the heat equations*

$$\frac{\partial S}{\partial t} = \Delta S \quad \text{and} \quad \frac{\partial T}{\partial t} = \Delta T$$

with $|T| < S$, then

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \frac{n}{2t} = \Delta \ln S + \frac{n}{2t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

Furthermore they generalized Hamilton's trace Harnack inequality [12] for the Ricci flow on surfaces with positive scalar curvature, and proved the following constrained linear trace Harnack inequality.

Theorem B (Chow-Hamilton [9]). *Let $g(t)$ be a solution to the Ricci flow on a closed surface M^2 with scalar curvature $R > 0$. If S and T are two solutions to*

$$\frac{\partial S}{\partial t} = \Delta S + RS \quad \text{and} \quad \frac{\partial T}{\partial t} = \Delta T + RT$$

with $|T| < S$, then

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \frac{1}{t} = \Delta \ln S + R + \frac{1}{t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

Recently, Y. Zheng and the author [28] generalized Theorem B and Chow's interpolated Harnack inequality [7] and proved the interpolated and constrained linear trace Harnack inequality.

Theorem C (Wu-Zheng [28]). *Let $g(t)$ be a solution to the ε -Ricci flow (1.1) on a closed surface M^2 with $R > 0$. If S and T are solutions to the following equations*

$$\frac{\partial S}{\partial t} = \Delta S + \varepsilon RS \quad \text{and} \quad \frac{\partial T}{\partial t} = \Delta T + \varepsilon RT$$

with $|T| < S$, then

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \frac{1}{t} = \Delta \ln S + \varepsilon R + \frac{1}{t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

In Theorem C, if we let $T \equiv 0$, then theorem recovers the Chow's interpolated Harnack inequality [7]. Chow's interpolation trick was also adapted to proving a matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow in the work of Ni [22].

Very recently, the author [27] derived an interesting interpolated Harnack inequality for the nonlinear parabolic equation (1.3), also extending Chow's interpolated Harnack inequality.

Theorem D (Wu [27]). *Let $(M, g(t))$, $t \in [0, \kappa)$, be a solution to the ε -Ricci flow (1.1) on a closed surface with $R > 0$. Let f be a positive solution to the nonlinear parabolic equation (1.3). Then for all time $t \in (0, \kappa)$,*

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \geq 0.$$

The main purpose of this paper is to generalize Theorems C and D, and establish an interpolated phenomenon for the nonlinear parabolic equation (1.3) under the ε -Ricci flow. We will see that this interpolated Harnack inequality is very similar to that of Theorem C. The main difference is that the parabolic equation of this paper possesses the additional nonlinear term: $f \ln f$. Hence in this case, the proof is a little subtle. Let S and T be solutions to the following nonlinear parabolic equations

$$(1.11) \quad \frac{\partial S}{\partial t} = \Delta S - S \ln S + \varepsilon RS$$

and

$$(1.12) \quad \frac{\partial T}{\partial t} = \Delta T - T \ln T + \varepsilon RT,$$

respectively, where Δ is the Laplacian of the metric moving under the ε -Ricci flow, with the property that initially

$$0 < c_0 S < T < S,$$

where c_0 is a free parameter, satisfying $0 < c_0 < 1$. Note that the above inequality is preserved along the ε -Ricci flow. In fact using (1.11) and (1.12), we compute the evolution equation of $h = T/S$:

$$(1.13) \quad \frac{\partial h}{\partial t} = \Delta h + 2\nabla h \cdot \nabla \ln S - h \ln h.$$

Applying the maximum principle to this equation, one can prove that the inequality: $c_0 < h < 1$ (and hence $c_0 S < T < S$) is preserved under the ε -Ricci flow.

Now we give the following interpolation theorem.

Theorem 1.1. *Let $g(t)$ be a solution to the ε -Ricci flow (1.1) on a closed surface M^2 with the initial scalar curvature satisfying*

$$(1.14) \quad R(g(0)) \geq -\frac{2 \ln c_0}{1 - c_0^2} - 1 > 0,$$

where c_0 is a free parameter, satisfying $0 < c_0 < 1$. If S and T are solutions to (1.11) and (1.12) with $0 < c_0 S < T < S$ (this condition preserved by the ε -Ricci flow), then

$$(1.15) \quad \frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \ln S + \frac{1}{t} = \Delta \ln S + \varepsilon R + \frac{1}{t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

Remark 1.1. *We would like to compare with Theorem C above. Theorem 1.1 can be regarded as a nonlinear version of Theorem C. In Theorem 1.1, if we remove the nonlinear terms: $S \ln S$ in (1.11) and $T \ln T$ in (1.12), then the term: $\ln S$ in (1.15) will disappear, and we can immediately get Theorem C under a slightly stronger scalar curvature assumption.*

Remark 1.2. *The theorem is also true on complete noncompact surface when the maximum principle can be used. For example, we can assume that the solution to the ε -Ricci flow is complete with the curvature and all the covariant derivatives being uniformly bounded, and $\Delta \ln S$ has a lower bound for all time t .*

As a consequence of Theorem 1.1, we have a classical Harnack inequality.

Theorem 1.2. *Let $g(t)$, $t \in (0, \kappa)$ be a solution to the ε -Ricci flow (1.1) on a closed surface M^2 with the initial scalar curvature satisfying (1.14). Let S and T be two solutions to (1.11) and (1.12) with $0 < c_0 S < T < S$. Assume that (x_1, t_1) and (x_2, t_2) , $0 < t_1 < t_2$, are two points in $M^2 \times (0, \kappa)$. Let*

$$\Gamma := \frac{1}{4} \inf_{\gamma} \int_{t_1}^{t_2} e^t \left(\left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{4}{t} \right) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2) . Then we have

$$e^{t_1} \ln S(x_1, t_1) < e^{t_2} \ln S(x_2, t_2) + \Gamma.$$

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.1. The proof nearly follows the proof of [28], which needs a lengthy but straight-forward computation and makes use of the parabolic maximum principle. In Section 3, using Theorem 1.1, we will prove Theorem 1.2 by the standard arguments.

2. Proof of Theorem 1.1

Under the ε -Ricci flow (1.1), we can compute that

$$(2.1) \quad \frac{\partial}{\partial t} \ln S = \Delta \ln S + |\nabla \ln S|^2 - \ln S + \varepsilon R$$

and

$$(2.2) \quad \frac{\partial}{\partial t} (\Delta) = \varepsilon R \Delta,$$

where the Laplacian Δ is acting on smooth functions. Now we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof follows from a direct computation and the parabolic maximum principle. Here we mainly follow the arguments of [28]. Note that the equation (1.3) is nonlinear. So our case is a little more complicated. Let

$$Q := \Delta \ln S + \varepsilon R = \frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \ln S,$$

where S is a positive solution to the equation (1.11). Following [28], using (2.1) and (2.2) we compute that

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta \left(\frac{\partial}{\partial t} \ln S \right) + \left(\frac{\partial}{\partial t} \Delta \right) \ln S + \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta (\Delta \ln S + |\nabla \ln S|^2 - \ln S + \varepsilon R) + \varepsilon R \Delta \ln S + \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta Q + \Delta |\nabla \ln S|^2 + (\varepsilon R - 1)Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon \frac{\partial R}{\partial t},\end{aligned}$$

where we used the equations (1.2), (2.1) and (2.2). Using the Bochner formula,

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta Q + 2|\nabla \nabla \ln S|^2 + 2\nabla \Delta \ln S \cdot \nabla \ln S + R|\nabla \ln S|^2 \\ &\quad + (\varepsilon R - 1)Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta Q + 2|\nabla \nabla \ln S|^2 + 2\nabla Q \cdot \nabla \ln S + R|\nabla \ln S|^2 \\ &\quad - 2\varepsilon \nabla R \cdot \nabla \ln S + (\varepsilon R - 1)Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta Q + 2\nabla Q \cdot \nabla \ln S - (\varepsilon R + 1)Q + 2 \left| \nabla \nabla \ln S + \frac{\varepsilon}{2} Rg \right|^2 \\ &\quad + R|\nabla \ln S - \varepsilon \nabla \ln R|^2 + \varepsilon R[\varepsilon(\Delta \ln R + R)] + \varepsilon R.\end{aligned}$$

Hence

$$(2.3) \quad \frac{\partial Q}{\partial t} \geq \Delta Q + 2\nabla Q \cdot \nabla \ln S - (\varepsilon R + 1)Q + 2 \left| \nabla \nabla \ln S + \frac{\varepsilon}{2} Rg \right|^2 + \varepsilon R[\varepsilon(\Delta \ln R + R)] + \varepsilon R.$$

Next by (1.13), the evolution equation of ∇h is given by

$$\begin{aligned}(2.4) \quad \frac{\partial}{\partial t}(\nabla h) &= \nabla \left(\frac{\partial h}{\partial t} \right) \\ &= \nabla [\Delta h + 2\nabla h \cdot \nabla \ln S - h \ln h] \\ &= \Delta \nabla h + 2\langle \nabla \nabla \ln S, \nabla h \rangle + 2\langle \nabla \ln S, \nabla \nabla h \rangle - \frac{R\nabla h}{2} - (1 + \ln h)\nabla h.\end{aligned}$$

Under the ε -Ricci flow, using (2.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla h|^2 &= 2\nabla h \left(\frac{\partial}{\partial t} \nabla h \right) - g^{ki}g^{lj} \frac{\partial}{\partial t} g_{kl} \nabla_i h \nabla_j h \\ &= 2\nabla h \left[\Delta \nabla h + 2\langle \nabla \nabla \ln S, \nabla h \rangle + 2\langle \nabla \ln S, \nabla \nabla h \rangle - \frac{R\nabla h}{2} - (1 + \ln h)\nabla h \right] \\ &\quad + \varepsilon R|\nabla h|^2 \\ &= \Delta |\nabla h|^2 - 2|\nabla \nabla h|^2 + 4\langle \nabla \nabla \ln S, \nabla h \nabla h \rangle + 2\langle \nabla \ln S, \nabla |\nabla h|^2 \rangle \\ &\quad + [(\varepsilon - 1)R - 2(1 + \ln h)]|\nabla h|^2.\end{aligned}$$

We also compute

$$\frac{\partial}{\partial t}(1 - h^2) = \Delta(1 - h^2) + 2\langle \nabla \ln S, \nabla(1 - h^2) \rangle + 2|\nabla h|^2 + 2h^2 \ln h.$$

Next we shall compute the evolution equation of $\frac{|\nabla h|^2}{1-h^2}$. Recall the following general result that if two functions E and F satisfy the heat equations of the form

$$\frac{\partial E}{\partial t} = \Delta E + A \quad \text{and} \quad \frac{\partial F}{\partial t} = \Delta F + B,$$

where A and B are some functions, then

$$\frac{\partial}{\partial t} \left(\frac{E}{F} \right) = \Delta \left(\frac{E}{F} \right) + \frac{2}{F^2} \langle \nabla E, \nabla F \rangle - \frac{2E}{F^3} |\nabla F|^2 + \frac{A}{F} - \frac{EB}{F^2}.$$

Applying this result to

$$E := |\nabla h|^2, \quad F := 1 - h^2,$$

$$B := 2\langle \nabla \ln S, \nabla(1 - h^2) \rangle + 2|\nabla h|^2 + 2h^2 \ln h$$

and

$$\begin{aligned} A := & -2|\nabla \nabla h|^2 + 4\langle \nabla \nabla \ln S, \nabla h \nabla h \rangle + 2\langle \nabla \ln S, \nabla |\nabla h|^2 \rangle \\ & + [(\varepsilon - 1)R - 2(1 + \ln h)] |\nabla h|^2, \end{aligned}$$

we get that

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{|\nabla h|^2}{1 - h^2} \right) = & \Delta \left(\frac{|\nabla h|^2}{1 - h^2} \right) + \frac{2\langle \nabla(1 - h^2), \nabla |\nabla h|^2 \rangle}{(1 - h^2)^2} - \frac{2|\nabla h|^2}{(1 - h^2)^3} |\nabla(1 - h^2)|^2 \\ & + \frac{1}{1 - h^2} \cdot [-2|\nabla \nabla h|^2 + 4\langle \nabla \nabla \ln S, \nabla h \nabla h \rangle] \\ & + \frac{2}{1 - h^2} \cdot \langle \nabla \ln S, \nabla |\nabla h|^2 \rangle + \frac{(\varepsilon - 1)R - 2(1 + \ln h)}{1 - h^2} |\nabla h|^2 \\ & - \frac{2|\nabla h|^2}{(1 - h^2)^2} \cdot [\langle \nabla \ln S, \nabla(1 - h^2) \rangle + |\nabla h|^2 + h^2 \ln h]. \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{|\nabla h|^2}{1 - h^2} \right) = & \Delta \left(\frac{|\nabla h|^2}{1 - h^2} \right) + 2 \left\langle \nabla \left(\frac{|\nabla h|^2}{1 - h^2} \right), \nabla \ln S \right\rangle \\ & - \frac{2}{(1 - h^2)^3} |2h \nabla h \nabla h + (1 - h^2) \nabla \nabla h|^2 \\ (2.5) \quad & + \frac{4}{1 - h^2} \langle \nabla \nabla \ln S, \nabla h \nabla h \rangle - \frac{2|\nabla h|^4}{(1 - h^2)^2} \\ & + \frac{(\varepsilon - 1)R - 2(1 + \ln h)}{1 - h^2} |\nabla h|^2 - \frac{2h^2 \ln h}{(1 - h^2)^2} |\nabla h|^2. \end{aligned}$$

Thus we define

$$P := Q - \frac{|\nabla h|^2}{1 - h^2} = \Delta \ln S + \varepsilon R - \frac{|\nabla h|^2}{1 - h^2}.$$

Combining (2.3) and (2.5), we conclude that

$$\begin{aligned}
\frac{\partial}{\partial t} P &\geq \Delta P + 2\nabla P \cdot \nabla \ln S - (\varepsilon R + 1)Q + \varepsilon R + 2 \left| \nabla \nabla \ln S + \frac{\varepsilon}{2} Rg \right|^2 \\
&\quad + \varepsilon R[\varepsilon(\Delta \ln R + R)] + \frac{2}{(1-h^2)^3} |2h\nabla h \nabla h + (1-h^2)\nabla \nabla h|^2 \\
&\quad - \frac{4}{1-h^2} \langle \nabla \nabla \ln S, \nabla h \nabla h \rangle + \frac{2|\nabla h|^4}{(1-h^2)^2} \\
&\quad + \frac{(1-\varepsilon)R + 2(1+\ln h)}{1-h^2} |\nabla h|^2 + \frac{2h^2 \ln h}{(1-h^2)^2} |\nabla h|^2 \\
&= \Delta P + 2\nabla P \cdot \nabla \ln S + 2 \left| \nabla \nabla \ln S + \frac{\varepsilon}{2} Rg - \frac{\nabla h \nabla h}{1-h^2} \right|^2 \\
&\quad + \varepsilon R[\varepsilon(\Delta \ln R + R)] + \frac{2}{(1-h^2)^3} |2h\nabla h \nabla h + (1-h^2)\nabla \nabla h|^2 \\
&\quad - (\varepsilon R + 1)Q + \varepsilon R + \frac{(1+\varepsilon)R + 2(1+\ln h)}{1-h^2} |\nabla h|^2 + \frac{2h^2 \ln h}{(1-h^2)^2} |\nabla h|^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
(2.6) \quad \frac{\partial}{\partial t} P &\geq \Delta P + 2\nabla P \cdot \nabla \ln S + P^2 - (\varepsilon R + 1)P + \varepsilon R[\varepsilon(\Delta \ln R + R)] \\
&\quad + \varepsilon R + \frac{|\nabla h|^2}{1-h^2} \left(R + 1 + \frac{2 \ln h}{1-h^2} \right),
\end{aligned}$$

where we used the elementary inequality

$$\left| \nabla \nabla \ln S + \frac{\varepsilon}{2} Rg - \frac{\nabla h \nabla h}{1-h^2} \right|^2 \geq \frac{1}{2} \left(\Delta \ln S + \varepsilon R - \frac{|\nabla h|^2}{1-h^2} \right)^2 = \frac{P^2}{2}.$$

Since $0 < c_0 < h < 1$ and the function $\frac{2 \ln h}{1-h^2}$ is increasing on $(0, 1)$, then

$$\frac{2 \ln h}{1-h^2} > \frac{2 \ln c_0}{1-c_0^2}.$$

By the assumption of the theorem, using the maximum principle, we can see that the inequality (1.14) still holds under the ε -Ricci flow. Hence

$$R + 1 + \frac{2 \ln h}{1-h^2} > R + 1 + \frac{2 \ln c_0}{1-c_0^2} > 0$$

for all time t . Therefore, since $(M, g(t))$ has positive scalar curvature, (2.6) becomes

$$\frac{\partial}{\partial t} P \geq \Delta P + 2\nabla P \cdot \nabla \ln S + P^2 - (\varepsilon R + 1)P + \varepsilon R[\varepsilon(\Delta \ln R + R)].$$

Adding $\frac{1}{t}$ to P yields

$$\begin{aligned} \frac{\partial}{\partial t} \left(P + \frac{1}{t} \right) &\geq \Delta \left(P + \frac{1}{t} \right) + 2\nabla \left(P + \frac{1}{t} \right) \cdot \nabla \ln S + \left(P + \frac{1}{t} \right) \left(P - \frac{1}{t} \right) \\ &\quad - (\varepsilon R + 1) \left(P + \frac{1}{t} \right) + \varepsilon R \left[\varepsilon(\Delta \ln R + R) + \frac{1}{t} \right]. \end{aligned}$$

Recall that the trace Harnack inequality for the ε -Ricci flow on a closed surface proved by B. Chow in [7] (see also Lemma 2.1 in [28]) implies

$$\frac{\partial \ln R}{\partial t} - \varepsilon |\nabla \ln R|^2 = \varepsilon(\Delta \ln R + R) \geq -\frac{1}{t},$$

since $g(t)$ has positive scalar curvature. Hence

$$\begin{aligned} \frac{\partial}{\partial t} \left(P + \frac{1}{t} \right) &\geq \Delta \left(P + \frac{1}{t} \right) + 2\nabla \left(P + \frac{1}{t} \right) \cdot \nabla \ln S + \left(P + \frac{1}{t} \right) \left(P - \frac{1}{t} \right) \\ &\quad - (\varepsilon R + 1) \left(P + \frac{1}{t} \right). \end{aligned}$$

It is clear to see that

$$P + 1/t > 0.$$

for very small positive t . Then applying the maximum principle to the above evolution formula, we conclude that

$$P + 1/t > 0$$

for all positive time t , and hence the desired theorem follows. \square

For Theorem 1.1, if we let $\varepsilon = 0$, then

Corollary 2.1. *Let M^2 be a closed surface with the scalar curvature satisfying (1.14). If S and T are solutions to*

$$\frac{\partial S}{\partial t} = \Delta S - S \ln S \quad \text{and} \quad \frac{\partial T}{\partial t} = \Delta T - T \ln T$$

with $0 < c_0 S < T < S$, then

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \ln S + \frac{1}{t} = \Delta \ln S + \frac{1}{t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

If we set

$$\bar{g} = \varepsilon^{-1} g \quad \text{and} \quad \alpha = \varepsilon^{-1}$$

in Theorem 1.1, then

$$\bar{\Delta} = \varepsilon \Delta \quad \text{and} \quad \bar{R} = \varepsilon R.$$

Hence Theorem 1.1 can be rephrased as follows:

Corollary 2.2. *Let $\bar{g}(t)$ be a solution to the Ricci flow on a closed surface M^2 with the initial scalar curvature satisfying*

$$\alpha \bar{R}(\bar{g}(0)) \geq -\frac{2 \ln c_0}{1 - c_0^2} - 1 > 0,$$

where α is a positive constant and c_0 is a free parameter, satisfying $0 < c_0 < 1$. If S and T are solutions to

$$\frac{\partial S}{\partial t} = \alpha \bar{\Delta} S - S \ln S + \bar{R} S \quad \text{and} \quad \frac{\partial T}{\partial t} = \alpha \bar{\Delta} T - T \ln T + \bar{R} T$$

with $0 < c_0 S < T < S$, then

$$\frac{\partial}{\partial t} \ln S - \alpha |\bar{\nabla} \ln S|^2 + \ln S + \frac{1}{t} = \alpha \bar{\Delta} \ln S + \bar{R} + \frac{1}{t} > \frac{\alpha |\bar{\nabla} h|^2}{1 - h^2},$$

where $h := T/S$.

3. PROOF OF THEOREM 1.2

In the rest of this paper, we will prove Theorem 1.2 by using Theorem 1.1. The proof is quite standard by integrating the inequality (1.15). We include it here for completeness.

Proof of Theorem 1.2. We pick a space-time path $\gamma(x, t)$ joining (x_1, t_1) and (x_2, t_2) with $t_2 > t_1 > 0$. Along γ , by Theorem 1.1 we have

$$\begin{aligned} \frac{d}{dt} \ln S(x, t) &= \frac{\partial}{\partial t} \ln S + \nabla \ln S \cdot \frac{d\gamma}{dt} \\ &> |\nabla \ln S|^2 - \ln S - \frac{1}{t} + \frac{|\nabla h|^2}{1 - h^2} + \nabla \ln S \cdot \frac{d\gamma}{dt} \\ &\geq -\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 - \ln S - \frac{1}{t}. \end{aligned}$$

Hence

$$\frac{d}{dt} (e^t \ln S(x, t)) > -e^t \left(\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{1}{t} \right).$$

Integrating this inequality from the time t_1 to t_2 yields

$$e^{t_1} \ln S(x_1, t_1) - e^{t_2} \ln S(x_2, t_2) < \int_{t_1}^{t_2} e^t \left(\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{1}{t} \right) dt.$$

By the definition of Γ , we finish the proof of Theorem 1.2. \square

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